Densities of distributions of solutions to delay stochastic differential equations with discontinuous initial data (Part I)

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Abstract

We have established an integration by parts formula involving Malliavinderevatives of solutions to the delay (functional) SDE's, See equation (1.1). The integration by parts formula which we have established is in fact an extension of the integration by parts formula to include delay SDE's as well as ordinary SDE's. The integration by parts formula which we have established can be used to extend the formulas in work by Bally and Talay to include delay SDE's as well as ordinary SDE's.

Keywords: Stochastic Differential Equations, Malliavin Calculus, Euler Scheme for delay SDE's, Integration by Parts, Densities of Distributions.

1. Introduction

In Chapter 1 of the Ph.D. thesis of Ahmed(15) we have proved the existence and uniqueness of a solution for certain types of delay (functional) stochastic differential equations (delay SDE's) with discontinuous initial data,see also(1),(9) , and the web cite www.sfde.math.siu.edu. See the delay SDE (1.1) in the present work. Here we establish an integration by parts formula involving solutions to such type of delay (functional) SDE's. The integration by parts formula which we establish can be used to extend the formulas in(2) and (3) to include delay SDE's as well as ordinary SDE's. In this work we also establish some other useful applications to delay SDE's. Generally speaking we can say that our work extends the first three chapters of the work by Norris to include delay SDE's as well as ordinary SDE's; see Theorems 2.3, 3.1 and 3.2 in (10). We will also show in a sequal paper to this work that the distribution of the solution process has smooth density. Also we will establish an integration by parts formula involving Malliavin derivatives of higher order.

1.1 Notations and Definitions

The following notations and definitions will be used throughout this work: $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space; T is a positive real number; $\{\mathcal{F}_t\}_{t\in[0,T]}$ is an increasing family of sub-σ algebras of $\mathcal{F}$, each of which contains all null subsets of Ω; $\mathbb{N}$ is the set of natural numbers; $W = (W^1, \dots, W^r)\colon [0,T] \times \Omega \to \mathbb{R}^r$ is a r-dimensional normalized Brownian motion. If X is a topological space, then $\mathcal{B}(X)$ denotes its Borel field. The symbol λ refers to the Lebesgue measure on $\mathbb{R}^d$, and $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$, $d \in \mathbb{N}$.

Let G be a Banach space and let $\mathcal{A}$ be a sub-σ algebra of $\mathcal{F}$ containing all subsets of measure zero in $\mathcal{F}$, then $\mathcal{L}^2(\Omega, \mathcal{A}, \mathbb{P}; G)$ denotes the space of all functions $f\colon \Omega \to G$ which are $\mathcal{A}$-$\mathcal{B}(G)$ measurable and are such that $\int_\Omega \|f\|_G^2 \, d\mathbb{P} < \infty$.

The symbol $L^2(\Omega, \mathcal{A}, \mathbb{P}; G)$ denotes the Banach space (with norm determined by $\|f\|_{L^2}^2 = \int_\Omega \|f(\omega)\|_G^2 \, d\mathbb{P}$) of all equivalence classes of functions $f\colon \Omega \to G$ which are $\mathcal{A}$-$\mathcal{B}(G)$ measurable and which are such that $\int_\Omega \|f\|_G^2 \, d\mathbb{P} < \infty$. The symbol $L(\mathbb{R}^m, \mathbb{R}^n)$ $(m, n \in \mathbb{N})$ denotes the space of all linear maps from $\mathbb{R}^m$ to $\mathbb{R}^n$. The symbol J refers to the interval $[-1,0)$, and $\mathcal{H}(J)$ or $\mathcal{B}(J)$ refers to the Borel field on J.

If $X\colon [-1,T] \times \Omega \to \mathbb{R}^d$ is a process, then for each $t \in [0,T]$ and $\omega \in \Omega$ we define the map: $X_t\colon \Omega \to \mathcal{L}^2(J, \mathbb{R}^d)$ by $X_t(\omega)(s) = X(t+s, \omega)$ for all $s \in J$ and almost all ω. For each $0 \le t \le T$ we write $\|(X(t), X_t)\|^2 = \|X(t)\|^2 + \|X_t\|^2$. Let the function V belong to $\mathcal{L}^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)$, θ belong to $\mathcal{L}^2(J \times \Omega, \mathcal{H}(J) \otimes \mathcal{F}_0, \lambda \otimes \mathbb{P}; \mathbb{R}^d)$, and for $\ell = 1,2,\dots,r$ let f ,g^ℓ be functions from $[0,T] \times \Omega \times \mathbb{R}^d \times \mathcal{L}^2(J, \mathbb{R}^d)$ to $\mathbb{R}^d$. Then a process $X\colon [-1,T] \times \Omega \to \mathbb{R}^d$ is called a solution of the delay SDE with integral form

$$X(t) = \begin{cases} V + \int_0^t f(u, X(u), X_u) du + \sum_{\ell=1}^{r} \int_0^t g^\ell(u, X(u), X_u) \, dW^\ell(u), & 0 \le t \le T, \\ \theta(t), & t \in J, \end{cases}$$

(1.1)

if

(i)X is $\mathcal{B}([0,T])\otimes\mathcal{F}$-$\mathcal{B}(\mathbb{R}^d)$ measurable;

(ii)For each $t\in[0,T]$, the process $X(t,\cdot)$ is $\mathcal{F}_t$-$\mathcal{B}(\mathbb{R}^d)$ measurable, and for each $t\in J$, the process $X(t,\cdot)$ is $\mathcal{F}_0$-$\mathcal{B}(\mathbb{R}^d)$ measurable;

(iii) $X\in\mathcal{L}^2([-1,T]\times\Omega,\mathcal{H}\times\mathcal{F},\lambda\times\mathbb{P};\mathbb{R}^d)$,

Xsatisfies the delay SDE (1.1).

The following conditions are sufficient for the existence of a unique solution to (1.1) (see [1] and [5]).

(i)$V\in\mathcal{L}^2(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{R}^d)$.

$(ii)\theta\in\mathcal{L}^2(J\times\Omega,\mathcal{H}\otimes\mathcal{F}_0,\lambda\otimes\mathbb{P},\mathbb{R}^d)$.

(iii) $f, g^\ell: [0,T]\times\Omega\times\mathbb{R}^d\times\mathcal{L}^2(J,\mathbb{R}^d)\to\mathbb{R}^d$ are such that

(a) $f$andg^ℓ are $\mathcal{B}([0,T])\otimes\mathcal{F}\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(L^2(J,\mathbb{R}^d))$-$\mathcal{B}(\mathbb{R}^d)$ measurable.

(b) For each $t\in[0,T]$, the stochastic variables $f(t,\cdot,\cdot,\cdot)$ and $g^\ell(t,\cdot,\cdot,\cdot)$ are $\mathcal{F}_t\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(\mathcal{L}^2(J,\mathbb{R}^d))$-$\mathcal{B}(\mathbb{R}^d)$ measurable.

(c)There exists a constant K and a function $\zeta\in\mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d)$ such that

$$|f(t,\omega,s,h)|+\sum_{\ell=1}^r\left|g^\ell(t,\omega,s,h)\right|\le K(|s|+\|h\|+|\zeta(\omega)|) \qquad (1.2)$$

for almost all ω and for all $t\in[0,T]$; $s\in\mathbb{R}^d$ and h belongs to $\mathcal{L}^2(J,\mathbb{R}^d)$.

(d) There exists a constant K' such that, for almost all ω,

$$|f(t,\omega,s,h_1)-f(t,\omega,u,h_2)|+\sum_{\ell=1}^r\left|g^\ell(t,\omega,s,h_1)-g^\ell(t,\omega,u,h_2)\right| \le K'(|s-u|+\|h_1-h_2\|) \qquad (1.3)$$

for all $t\in[0,T]$; for all $s,u\in\mathbb{R}^d$, and for all $h_1,h_2\in\mathcal{L}^2(J,\mathbb{R}^d)$.

2 Integration by parts formula

In this section we shall extend the first three chapters of the work of Norris to include not only Stochastic Differential Equations (SDE's), but also Delay SDE's of type (1.1). For $(X(0),X_0)=(x,\xi)\in\mathbb{R}^d\times L^2(J,\mathbb{R}^d)$, let $v\mapsto D^vX^{x,\xi}(t)$, be the Malliavin derivative of the solution process

$X^{x,\xi}(t)$. We write $D^v X_t^{x,\xi}(\vartheta) = D^v X^{x,\xi}(t+\vartheta)$ ($t \in [0,T]$, $\vartheta \in J = [-1,0)$) for its time delay. In the following definition we give a precise definition of the Malliavin derivative of a real-valued functional F of Brownian motion.

1 Definition. Let $F((W(s))_{0\le s\le T})$ be a functional of r-dimensional Brownian motion, and let $v(t) = (v^1(t), \dots, v^r(t))^* = \begin{pmatrix} v^1(t) \\ \vdots \\ v^r(t) \end{pmatrix}$ be a deterministic vector-valued function in $L^2([0,T], \mathbb{R}^r \otimes \mathbb{R}^d)$. Then $D^v F((W(s))_{0\le s\le T})$ is given by the limit:

$$D^v F((W(s))_{0\le s\le T}) = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\left(F\left(\left(W(s) + \varepsilon\int_0^s v(\sigma)d\sigma\right)_{0\le s\le s}\right) - F((W(s))_{0\le s\le t})\right). \quad (2.1)$$

The mapping $v \mapsto D^v F((W(s))_{0\le s\le T})$ is a linear map (functional) from the space $L^2([0,T], \mathbb{R}^r \otimes \mathbb{R}^d)$ to $\mathbb{R}$. Here $\mathbb{R}^r \otimes \mathbb{R}^d$ denotes the space of all $r \times d$-matrices (r rows, d columns).

Notice that, for $v(t) = (v^1(t), \dots, v^r(t))^\tau = \begin{pmatrix} v^1(t) \\ \vdots \\ v^r(t) \end{pmatrix}$ be a deterministic matrix-valued function in $L^2([0,T], \mathbb{R}^r \otimes \mathbb{R}^d)$, $U^v(t)$ can be considered as a $d \times d$-matrix where each entry is an $\mathbb{R}$-valued adapted stochastic process; U_t^v can be considered as a $d \times d$-matrix where each entry is an $L^2(J,\mathbb{R})$-valued adapted stochastic process. If $M = (m_{jk})_{1\le j\le d,\, 1\le k\le r}$ is a real $d \times r$ matrix, then $M^\tau = (m_{kj})_{1\le k\le r,\, 1\le j\le d}$ denotes its transposed: it is $r \times d$ matrix with entries m_{kj}. The process $D^v X_t^{x,\xi}(\cdot)$ satisfies the following delay stochastic differential equation:

$$\begin{aligned} dD^v X_t(\vartheta) &= dD^v X(t+\vartheta) \\ &= \left(\frac{\partial f}{\partial x}(t+\vartheta, X(t+\vartheta), X_{t+\vartheta}) D^v X(t+\vartheta) \right. \\ &\quad \left. + \int_J \frac{\partial f}{\partial \xi}(t+\vartheta, X(t+\vartheta), X_{t+\vartheta})(\varphi) D^v X_{t+\vartheta}(\varphi)\, d\varphi \right) dt \\ &\quad + \sum_{\ell=1}^r \frac{\partial g^\ell}{\partial x}(t+\vartheta, X(t+\vartheta), X_{t+\vartheta}) D^v X(t+\vartheta) dW^\ell(t+\vartheta) \\ &\quad + \sum_{\ell=1}^r \int_J \frac{\partial g^\ell}{\partial \xi}(t+\vartheta, X(t+\vartheta), X_{t+\vartheta})(\varphi) D^v X_{t+\vartheta}(\varphi)\, d\varphi\, dW^\ell(t+\vartheta) \\ &\quad + \sum_{\ell=1}^r g^\ell(t+\vartheta, X(t+\vartheta), X_{t+\vartheta}) v^\ell(t+\vartheta, X(t+\vartheta), X_{t+\vartheta}) dt, \quad (2.2) \end{aligned}$$

where ϑ belongs to J. If $t+\vartheta$ belongs to J we replace $t+\vartheta$ with 0in(2.2). If $\vartheta = 0$ we obtain the delay stochastic differential equation for the process $D^v X(t)$:

$$\begin{aligned} dD^{v}X(t) \\ = \left(\frac{\partial f}{\partial x}(t, X(t), X_t)D^{v}X(t) + \int_J \frac{\partial f}{\partial \xi}(t, X(t), X_t)(\vartheta)D^{v}X_t(\vartheta)d\vartheta\right)dt \\ + \sum_{\ell=1}^{r}\left(\frac{\partial g^{\ell}}{\partial x}(t, X(t), X_t)D^{v}X(t) + \int_J \frac{\partial g^{\ell}}{\partial \xi}(t, X(t), X_t)(\vartheta)D^{v}X_t(\vartheta)d\vartheta\right)dW^{\ell}(t) \\ + \sum_{\ell=1}^{r} g^{\ell}(t, X(t), X_t)v^{\ell}(t, X(t), X_t)dt. \end{aligned} \tag{2.3}$$

We also write $U_{11}^{x,\xi}(t) = \frac{\partial}{\partial x}X^{x,\xi}(t)$, and $U_{12}^{x,\xi}(t) = \frac{\partial}{\partial \xi}X^{x,\xi}(t)$. In addition, we write $U_{21}^{x,\xi}(t) = \frac{\partial}{\partial x}X_t^{x,\xi} = U_{11,t}^{x,\xi}$ (the delay of $U_{11}^{x,\xi}(t)$), and $U_{22}^{x,\xi}(t) = \frac{\partial}{\partial \xi}X_t^{x,\xi} = U_{12,t}^{x,\xi}$, the delay of the process $U_{12}^{x,\xi}(t)$. The matrix $U_{11}^{x,\xi}(t)$ can be identified with an operator from $\mathbb{R}^d$ to itself, the matrix $U_{12}^{x,\xi}(t)$ can be considered as an linear mapping from $L^2(J, \mathbb{R}^d)$ to $\mathbb{R}^d$, the matrix $U_{21}^{x,\xi}(t)$ as a mapping from $\mathbb{R}^d$ to $L^2(J, \mathbb{R}^d)$, and, finally, $U_{22}^{x,\xi}(t)$ as a mapping from $L^2(J, \mathbb{R}^d)$ to itself. Notice that $U_{11}^{x,\xi}(t)$ can be considered as $d \times d$-matrix where each entry is an $\mathbb{R}$-valued adapted stochastic process; $U_{12}^{x,\xi}(t)$ can be considered as $d \times d$-matrix where each entry is an $L^2(J, \mathbb{R})$-valued adapted stochastic process. To be precise, write the solution process as a d-vector $X^{x,\xi}(t) = \left(X_1^{x,\xi}(t), \dots, X_d^{x,\xi}(t)\right)$, and consider the mapping ($1 \le j,\, k \le d$)

$$\xi_k \to X_j^{x,(\xi_1,\dots,\xi_{k-1},\xi_k,\xi_{k+1},\dots,\xi_d)}(t), \tag{2.4}$$

which is a mapping from $L^2(J, \mathbb{R})$ to $\mathbb{R}$, and where each variable ξ_ℓ, $l \neq k$, is a fixed function in $L^2(J, \mathbb{R})$. The derivative of the function in (2.4) can be considered as a continuous linear functional on $L^2(J, \mathbb{R})$. Therefore it can be represented as an inner-product with a function in $L^2(J, \mathbb{R})$, which is denoted by $\frac{\partial X_j^{x,\xi}(t)}{\partial \xi_k}$. Consequently, we write

$$\begin{aligned} \frac{\partial X_j^{x,\xi}(t)}{\partial \xi_k}(\eta) &= \lim_{h\to 0}\frac{X_j^{x,(\xi_1,\dots,\xi_{k-1},\xi_k+h\eta,\xi_{k+1},\dots,\xi_d)}(t) - X_j^{x,(\xi_1,\dots,\xi_{k-1},\xi_k,\xi_{k+1},\dots,\xi_d)}(t)}{h} \\ &= \int_J \eta(\varphi)\frac{\partial X_j^{x,\xi}(t)}{\partial \xi_k}(\varphi)d\varphi, \quad \eta \in L^2(J, \mathbb{R}). \end{aligned} \tag{2.5}$$

The process $t \mapsto U_{12}^{x,\xi}(t)$ can be identified with the matrix $t \mapsto \left(\frac{\partial X_j^{x,\xi}(t)}{\partial \xi_k}\right)_{j,k=1}^{d}$, where each entry $\frac{\partial X_j^{x,\xi}(t)}{\partial \xi_k}$ is an $L^2(J, \mathbb{R})$-function. The process $t \mapsto \frac{\partial X_{j,t}^{x,\xi}}{\partial \xi_k}$ then stands for the $L^2(J \times J, \mathbb{R})$-valued process given by

$$(t,(\vartheta,\varphi)) \mapsto \frac{\partial X_j^{x,\xi}(t+\vartheta)}{\partial \xi_k}(\varphi), t \geq 0, \ \ (\vartheta,\varphi) \in J \times J.$$

The process $t \mapsto U_{22}(t) = U_{12,t}$ can be identified with the $d \times d$-matrix

$$(t,(\vartheta,\varphi)) \mapsto \left(\frac{\partial X_j^{x,\xi}(t+\vartheta)}{\partial \xi_k}(\varphi)\right)_{j,k=1}^{d}, \ \ t \geq 0, \ \ (\vartheta,\varphi) \in J \times J.$$

The matrix-valued process $U_{11}^{x,\xi}(t)$ satisfies the following delay stochastic differential equation:

$$\begin{aligned} dU_{11}(t) &= \left(\frac{\partial f}{\partial x}(t,X(t),X_t)U_{11}(t) + \int_J \frac{\partial f}{\partial \xi}(t,X(t),X_t)(\vartheta)U_{11,t}(\vartheta)d\vartheta\right)dt \\ &+ \sum_{\ell=1}^{r}\left(\frac{\partial g^\ell}{\partial x}(t,X(t),X_t)U_{11}(t) + \int_J \frac{\partial g^\ell}{\partial \xi}(t,X(t),X_t)(\vartheta)U_{11,t}(\vartheta)d\vartheta\right)dW^\ell(t), \end{aligned} \tag{2.6}$$

where $U_{11,t} = \frac{\partial}{\partial x}X_t^{x,\xi}$. Here $\int_J \frac{\partial f}{\partial \xi}(t,X(t),X_t)(\vartheta)U_{11,t}(\vartheta)d\vartheta$ is a $d \times d$ matrix with real-valued entries

$$\sum_{\ell=1}^{d}\int_J \frac{\partial f_j}{\partial \eta_\ell}\left(t,X^{x,\xi}(t),X_t^{x,\xi}\right)(\vartheta)\frac{\partial X_{\ell,t}^{x,\xi}}{\partial x_k}(\vartheta)d\vartheta.$$

Similarly the matrix $\int_J \frac{\partial g^\ell}{\partial \xi_k}(t,X(t),X_t)(\vartheta)U_{11,t}(\vartheta)d\vartheta$ has real-valued entries

$$\sum_{m=1}^{d}\int_J \frac{\partial g_j^\ell}{\partial \eta_m}\left(t,X^{x,\xi}(t),X_t^{x,\xi}\right)(\vartheta)\frac{\partial X_{m,t}^{x,\xi}}{\partial x_k}(\vartheta)d\vartheta.$$

The process $U_{12}^{x,\xi}(t)$ satisfies the following delay stochastic differential equation:

$$\begin{aligned} &dU_{12}(t) \\ &= \left(\frac{\partial f}{\partial x}(t,X(t),X_t)U_{12}(t) + \int_J \frac{\partial f}{\partial \xi}(t,X(t),X_t)(\vartheta)U_{12,t}(\vartheta)d\vartheta\right)dt \\ &+ \sum_{\ell=1}^{r}\left(\frac{\partial g^\ell}{\partial x}(t,X(t),X_t)U_{12}(t) + \int_J \frac{\partial g^\ell}{\partial \xi}(t,X(t),X_t)(\vartheta)U_{12,t}(\vartheta)d\vartheta\right)dW^\ell(t), \end{aligned} \tag{2.7}$$

where $U_{12,t} = \frac{\partial}{\partial \xi}X_t^{x,\xi}$. Similarly the matrix $\int_J \frac{\partial f}{\partial \xi}(t,X(t),X_t)(\vartheta)U_{12,t}(\vartheta)d\vartheta$ is a $d \times d$ matrix with has real-valued entries

$$\sum_{m=1}^{d}\int_J \frac{\partial f_j}{\partial \eta_m}\left(t,X^{x,\xi}(t),X_t^{x,\xi}\right)(\vartheta)\frac{\partial X_{m,t}^{x,\xi}}{\partial \xi_k}(\vartheta)d\vartheta.$$

Similarly the matrix $\int_J \frac{\partial g^\ell}{\partial \xi}(t, X(t), X_t)(\vartheta)U_{12,t}(\vartheta)d\vartheta$ has real-valued entries

$$\sum_{m=1}^{d} \int_J \frac{\partial g_j^\ell}{\partial \eta_m}\left(t, X^{x,\xi}(t), X_t^{x,\xi}\right)(\vartheta) \frac{\partial X_{m,t}^{x,\xi}}{\partial \xi_k}(\vartheta)d\vartheta.$$

We shall call the solutions to the equations (2.6) and (2.7), the space flow and the L^2-flow respectively. For brevity we suppress the dependence on the initial condition $(x, \xi) \in \mathbb{R}^d \times L^2(J, \mathbb{R}^d)$ of the matrices $U_{jk}(t) = U_{jk}^{x,\xi}(t)$, $j, k = 1, 2$. We also write $U(t, \vartheta) = \begin{pmatrix} U_{11}^{x,\xi}(t) & U_{12}^{x,\xi}(t) \\ U_{11}^{x,\xi}(t+\vartheta) & U_{12}^{x,\xi}(t+\vartheta) \end{pmatrix}$. Then the pair $\left(U_{11}^{x,\xi}(t), U_{12}^{x,\xi}(t)\right)$ satisfies the following delay stochastic differential equation:

$$d\left(U_{11}(t), U_{12}(t)\right) = \int_J \left(\frac{\partial f}{\partial x}(t)dt + \sum_{\ell=1}^{r} \frac{\partial g^\ell}{\partial x}(t)dW^\ell(t), \frac{\partial f}{\partial \xi}(t,\vartheta)dt + \sum_{\ell=1}^{r} \frac{\partial g^\ell}{\partial \xi}(t,\vartheta)dW^\ell(t) \right) U(t,\vartheta)d\vartheta. \quad (2.8)$$

Here, and in the sequel, we write $f(t)$ and $g^\ell(t)$ instead of $f\left(t, X^{x,\xi}(t), X_t^{x,\xi}\right)$ and $g^\ell\left(t, X^{x,\xi}(t), X_t^{x,\xi}\right)$ respectively. For a concise formulation of the stochastic differential equation for the matrix-valued process $(U(t): t \geq)$ and its inverse we introduce the following *stochastic differentials*:

$$h_x(t) = \frac{\partial f}{\partial x}(t)dt + \sum_{\ell=1}^{r} \frac{\partial g^\ell}{\partial x}(t)dW^\ell(t); \quad (2.9)$$

$$h_\xi(t) = \frac{\partial f}{\partial \xi}(t)dt + \sum_{\ell=1}^{r} \frac{\partial g^\ell}{\partial \xi}(t)\,dW^\ell(t) \quad (2.10)$$

$$h_\xi(t,\vartheta) = \frac{\partial f}{\partial \xi}(t,\vartheta)dt + \sum_{\ell=1}^{r} \frac{\partial g^\ell}{\partial \xi}(t,\vartheta)dW^\ell(t) \quad (2.11)$$

Notice that $h_\xi(t,\cdot)$ is a process of stochastic differentials, which can be considered as a differential with coefficients in $L^2(J, \mathbb{R}^d)$. It is mentioned that in the present paper we will mainly be using the process $U_{11}^{x,\xi}(t)$; in fact as of now we write $U(t)$ instead of $U_{11}^{x,\xi}(t)$.

The process $U(t)$ satisfies the following Delay Stochastic Differential Equation:

$$dU(t) = h_x(t) \circ U(t) + \int_J h_\xi(t,\vartheta) \circ U_t(\vartheta)d\vartheta, \quad (2.12)$$

and its delay $(U_t: t \geq 0)$ satisfies:

$$dU_t(\cdot) = h_{x,t}(\cdot) \circ U_t(\cdot) + \int_J h_{\xi,t}(\cdot,\varphi) \circ U_{t+\cdot}(\varphi)d\varphi. \quad (2.13)$$

where, for $\vartheta \in J$, ([E: SDEU delay1]) is equivalent to

$$dU(t+\vartheta) = h_x(t+\vartheta) \circ U(t+\vartheta) + \int_J h_\xi\,(t+\vartheta)(\varphi)U_{t+\vartheta}(\varphi)d\varphi. \quad (2.14)$$

Here, symbols like $h_x(t) \circ U(t)$ and $h_\xi(t) \circ U_t$ have to be interpreted as *Stratonovich* differentials. In other words the integrals $\int_0^t h_x\,(s) \circ U(s)$ and $\int_0^t h_\xi\,(s) \circ U_s$ must be treated as Stratonovich integrals. Put $F(t) = U(t)V(t)$, and suppose that $F(0) = I$ and

$$dF(t) = h_x(t) \circ F(t) - F(t) \circ h_x(t) + \int_J h_\xi\,(t,\varphi) \circ F_t(\varphi)d\varphi - \int_J F_t\,(\varphi) \circ h_\xi(t,\varphi)d\varphi,$$

with $F(0) = I$. Then $F(t) = U(t)V(t) = I$. This follows from the uniqueness part of the theory of stochastic differential equations. This means that if the process $(U(t): t \geq 0)$ is given, then $V(t)$ is its inverse.

Let us recall the following equations:

$$U(t)V(t) = U(0)V(0) = I, \text{ and thus} \quad (2.16)$$
$$0 = \quad d[U(\cdot)V(\cdot)](t) = dU(t)V(t) + U(t)dV(t) + d < U(\cdot),V(\cdot) > (t). \quad (2.17)$$

Then by multiplying equation (2.17) by $V(t)$ and using the fact that $U(t)V(t) = I$ we obtain

$$V(t)dU(t)V(t) + dV(t) + V(t)d < U(\cdot),V(\cdot) > (t) = 0 \quad (2.18)$$

We also recall the already given equation

$$dU(t) = h_x(t)U(t) + \int_J h_\xi\,(t,\vartheta)U_t(\vartheta)d\vartheta. \quad (2.19)$$

Next we observe that equations (2.18) and (2.19) and $U(t)V(t) = I$ enable us to rewrite $d < U(\cdot),V(\cdot) > (t)$ in the following form

$$V(t)h_x(t) + V(t)\int_J h_\xi\,(t,\vartheta)U_t(\vartheta)d\vartheta V(t) + dV(t) + V(t)d < U(\cdot),V(\cdot) > (t) = 0. \quad (2.20)$$

Observe that equations (2.18) and (2.20) are equivalent. Then by looking at the martingale parts of the SDE's (2.19) and (2.20) we can compute the covariance process in (2.20) and get

$$d < U(\cdot), V(\cdot) > (t)$$

$$= -\sum_{\ell=1}^{r} \frac{\partial g^\ell}{\partial x}(t)U(t)V(t)\frac{\partial g^\ell}{\partial x}(t)d\,t - \sum_{\ell=1}^{r} \frac{\partial g^\ell}{\partial x}(t)U(t)V(t)\int_J \frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta\, V(t)d\,t$$

$$-\sum_{\ell=1}^{r} \int_J \frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta\, V(t)\frac{\partial g^\ell}{\partial x}(t)d\,t$$

$$-\sum_{\ell=1}^{r} \int_J \frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta\, V(t)\int_J \frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta V(t)d\,t$$

$$= -\sum_{\ell=1}^{r} \left(\frac{\partial g^\ell}{\partial x}(t)\right)^2 d\,t - \sum_{\ell=1}^{r} \frac{\partial g^\ell}{\partial x}(t)\int_J \frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta\, V(t)d\,t$$

$$-\sum_{\ell=1}^{r} \int_J \frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta\, V(t)\frac{\partial g^\ell}{\partial x}(t)d\,t - \sum_{\ell=1}^{r} \int_J \left(\frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta\, V(t)\right)^2 d\,t$$

$$= -\sum_{\ell=1}^{r} \left(\frac{\partial g^\ell}{\partial x}(t) + \int_J \frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta\, V(t)\right)^2 dt. \qquad (2.21)$$

It follows that $V(t)$ satisfies the following stochastic differential equation:

$$dV(t) = -V(t)h_x(t) - V(t)\left(\int_J h_\xi(t,\vartheta)U_t(\vartheta)d\vartheta\right)V(t)$$

$$+V(t)\sum_{\ell=1}^{r} \left(\frac{\partial g^\ell}{\partial x}(t) + \int_J \frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta V(t)\right)^2 dt. \qquad (2.22)$$

Remark: Now we try the following Delay Stochastic Differential Equation:

$$dV(t) = -V(t)h_x(t) - \int_J V_t\,(\vartheta)h_\xi(t,\vartheta)d\vartheta + V(t)A(t)d\,t + \int_J V_t\,(\vartheta)B(t,\vartheta)d\vartheta\, d\,t. \quad (2.23)$$

Suppose that a process like $V(t)$, $t \geq 0$, exists; i.e. suppose that $U(t)V(t) = V(t)U(t) = I$ (on an appropriate Hilbert space). Then a combination of (2.23), (2.19), (2.20), (2.21), and (2.10) yields:

$$
\begin{aligned}
&V(t)h_x(t) + V(t)\int_J h_\xi\,(t,\vartheta)U_t(\vartheta)d\vartheta V(t) - V(t)h_x(t) - \int_J V_t\,(\vartheta)h_\xi(t,\vartheta)d\vartheta \\
&+V(t)A(t)dt + \int_J V_t\,(\vartheta)B(t,\vartheta)d\vartheta\,dt \\
&-V(t)\sum_{\ell=1}^{r}\frac{\partial g^\ell}{\partial x}(t)U(t)V(t)\frac{\partial g^\ell}{\partial x}(t)d\,t \\
&-V(t)\sum_{\ell=1}^{r}\frac{\partial g^\ell}{\partial x}(t)U(t)V(t)\int_J\frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta V(t)d\,t \\
&-V(t)\sum_{\ell=1}^{r}\int_J\frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta\,V(t)\frac{\partial g^\ell}{\partial x}(t)d\,t \\
&-V(t)\sum_{\ell=1}^{r}\int_J\frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta\,V(t)\int_J\frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta\,V(t)d\,t \\
&= V(t)\int_J\frac{\partial f}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta\,V(t)\,dt + \sum_{\ell=1}^{r}V\,(t)\int_J\frac{\partial g^\ell}{\partial \xi}(t,\vartheta)d\,W^\ell(t)U_t(\vartheta)d\vartheta\,V(t) \\
&-\int_J V_t\,(\vartheta)\frac{\partial f}{\partial \xi}(t,\vartheta)d\vartheta\;\;dt - \sum_{\ell=1}^{r}\int_J V_t\,(\vartheta)\frac{\partial g^\ell}{\partial \xi}(t,\vartheta)d\vartheta\,d\,W^\ell(t) \\
&+V(t)A(t)dt + \int_J V_t\,(\vartheta)B(t,\vartheta)d\vartheta\,dt \\
&-V(t)\sum_{\ell=1}^{r}\left(\frac{\partial g^\ell}{\partial x}(t)\right)^2 d\,t - V(t)\sum_{\ell=1}^{r}\frac{\partial g^\ell}{\partial x}(t)\int_J\frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta\,V(t)d\,t \\
&-V(t)\sum_{\ell=1}^{r}\int_J\frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta\,V(t)\frac{\partial g^\ell}{\partial x}(t)d\,t \\
&-V(t)\sum_{\ell=1}^{r}\left(\int_J\frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta\,V(t)\right)^2 d\,t = 0.
\end{aligned}
\tag{2.24}
$$

A possible choice for $A(t)$ could be

$$A(t) = -\int_J \frac{\partial f}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta\, V(t) \qquad (2.25)$$
$$+\sum_{\ell=1}^{r}\left(\frac{\partial g^\ell}{\partial x}(t)\right)^2+\sum_{\ell=1}^{r}\frac{\partial g^\ell}{\partial x}(t)\int_J \frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta V(t)$$
$$+\sum_{\ell=1}^{r}\int_J \frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta\, V(t)\frac{\partial g^\ell}{\partial x}(t)+\sum_{\ell=1}^{r}\left(\int_J \frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta\, V(t)\right)^2$$
$$= -\int_J \frac{\partial f}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta\, V(t)+\sum_{\ell=1}^{r}\left(\frac{\partial g^\ell}{\partial x}(t)+\int_J \frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta\, V(t)\right)^2,$$

and for $B(t)$ it could be

$$B(t) = \frac{\partial f}{\partial \xi}(t). \qquad (2.26)$$

In order to achieve (2.24) we are then lead to the equation

$$\sum_{\ell=1}^{r} V(t)\int_J \frac{\partial g^\ell}{\partial \xi}(t,\vartheta)d\,W^\ell(t)U_t(\vartheta)d\vartheta V(t) - \sum_{\ell=1}^{r}\int_J V_t(\vartheta)\frac{\partial g^\ell}{\partial \xi}(t,\vartheta)d\vartheta d\,W^\ell(t) = 0. \qquad (2.27)$$

Equation (2.27) is equivalent to the following one:

$$\sum_{\ell=1}^{r}\int_J V(t)\frac{\partial g^\ell}{\partial \xi}(t,\vartheta)d\,W^\ell(t)U_t(\vartheta)d\vartheta V(t) = \sum_{\ell=1}^{r}\int_J V_t(\vartheta)\frac{\partial g^\ell}{\partial \xi}(t,\vartheta)d\vartheta d\,W^\ell(t). \qquad (2.28)$$

If (2.28) were true then the choice (2.25) and (2.26) of the respective processes $A(t)$ and $B(t)$ would result in the following equation for $V(t)$:

$$
\begin{aligned}
dV(t) = \quad & -V(t)h_x(t) - \int_J V_t(\vartheta)h_\xi(t,\vartheta)d\vartheta - V(t)\int_J \frac{\partial f}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta V(t)\,dt \\
& +V(t)\sum_{\ell=1}^{r}\left(\frac{\partial g^\ell}{\partial x}(t)\right)^2 dt + V(t)\sum_{\ell=1}^{r}\frac{\partial g^\ell}{\partial x}(t)\int_J \frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta\, V(t)\,dt \\
& +V(t)\sum_{\ell=1}^{r}\int_J \frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta\, V(t)\frac{\partial g^\ell}{\partial x}(t)\,dt \\
& +V(t)\sum_{\ell=1}^{r}\int_J\left(\frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta\, V(t)\right)^2 dt + \int_J V_t(\vartheta)\frac{\partial f}{\partial \xi}(t,\vartheta)d\vartheta\,dt \\
= \quad & -V(t)\frac{\partial f}{\partial x}(t)dt - V(t)\sum_{\ell=1}^{r}\frac{\partial g^\ell}{\partial x}(t)dW^\ell(t) \\
& -\sum_{\ell=1}^{r}\int_J V_t(\vartheta)\frac{\partial g^\ell}{\partial \xi}(t,\vartheta)d\vartheta\, dW^\ell(t) - V(t)\int_J \frac{\partial f}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta\, V(t)dt \\
& +V(t)\sum_{\ell=1}^{r}\left(\frac{\partial g^\ell}{\partial x}(t)\right)^2 dt + V(t)\sum_{\ell=1}^{r}\frac{\partial g^\ell}{\partial x}(t)\int_J \frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta V(t)\,dt \\
& +V(t)\sum_{\ell=1}^{r}\int_J \frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta\, V(t)\frac{\partial g^\ell}{\partial x}(t)\,dt \\
& +V(t)\sum_{\ell=1}^{r}\left(\int_J \frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta\, V(t)\right)^2 dt \\
= \quad & -V(t)\frac{\partial f}{\partial x}(t)dt - V(t)\sum_{\ell=1}^{r}\frac{\partial g^\ell}{\partial x}(t)dW^\ell(t) \\
& -\sum_{\ell=1}^{r}\int_J V_t(\vartheta)\frac{\partial g^\ell}{\partial \xi}(t,\vartheta)d\vartheta\, dW^\ell(t) - V(t)\int_J \frac{\partial f}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta V(t)\,dt \\
& +V(t)\sum_{\ell=1}^{r}\left(\frac{\partial g^\ell}{\partial x}(t) + \int_J \frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)d\vartheta V(t)\,dt\right)^2. \quad (2.29)
\end{aligned}
$$

Consequently, equation (2.22) and equation (2.29) are equivalent, provided condition (2.28) is satisfied.

Remark. Next we get the delay SDE for V_t as follows:

$$
\begin{aligned}
dV_t = {} & -V_t h_{x,t} - \int_J V_{t+\cdot}(\vartheta) h_{\xi,t}(\vartheta)\, d\vartheta - V_t \int_J \frac{\partial f_{t+\cdot}}{\partial \xi}(\vartheta) U_{t+\cdot}(\vartheta)\, d\vartheta V_t\, dt \\
& + V_t \sum_{\ell=1}^{r} \left(\frac{\partial g_t^\ell}{\partial x}\right)^2 dt + V_t \sum_{\ell=1}^{r} \frac{\partial g_t^\ell}{\partial x} \frac{\partial g_{t+\cdot}^\ell}{\partial \xi}(\vartheta) U_{t+\cdot}(\vartheta) d\vartheta V_t\, dt \\
& + V_t \sum_{\ell=1}^{r} \int_J \frac{\partial g_{t+\cdot}^\ell}{\partial \xi}(\vartheta) U_{t+\cdot}(\vartheta)\, d\vartheta V_t \frac{\partial g_t^\ell}{\partial x}\, dt + V_t \sum_{\ell=1}^{r} \int_J \left(\frac{\partial g_{t+\cdot}^\ell}{\partial \xi}(\vartheta) U_{t+\cdot}(\vartheta) d\vartheta V_t\right)^2 dt \\
& + V_{t+\cdot}(\vartheta) \frac{\partial f_{t+\cdot}}{\partial \xi}(\vartheta)\, d\vartheta\, dt \\
= {} & -V_t \frac{\partial f_t}{\partial x} dt - V_t \sum_{\ell=1}^{r} \frac{\partial g_t^\ell}{\partial x} dW_t^\ell \\
& - \sum_{\ell=1}^{r} \int_J V_{t+\cdot}(\vartheta) \frac{\partial g_{t+\cdot}^\ell}{\partial \xi}(\vartheta) d\vartheta dW_t^\ell - V_t \int_J \frac{\partial f_{t+\cdot}}{\partial \xi}(\vartheta) U_{t+\cdot}(\vartheta) d\vartheta V_t dt \\
& + V_t \sum_{\ell=1}^{r} \left(\frac{\partial g_t^\ell}{\partial x}\right)^2 dt + V_t \sum_{\ell=1}^{r} \frac{\partial g_t^\ell}{\partial x} \int_J \frac{\partial g_{t+\cdot}^\ell}{\partial \xi}(\vartheta) U_{t+\cdot}(\vartheta) d\vartheta V_t\, dt \\
& + V_t \sum_{\ell=1}^{r} \int_J \frac{\partial g_{t+\cdot}^\ell}{\partial \xi}(\vartheta) U_{t+\cdot}(\vartheta) d\vartheta V_t \frac{\partial g_t^\ell}{\partial x}\, dt + V_t \sum_{\ell=1}^{r} \int_J \left(\frac{\partial g_{t+\cdot}^\ell}{\partial \xi}(\vartheta) U_{t+\cdot}(\vartheta) d\vartheta V_t\right)^2 dt \\
= {} & -V_t \frac{\partial f_t}{\partial x} dt - V_t \sum_{\ell=1}^{r} \frac{\partial g_t^\ell}{\partial x} dW_t^\ell \\
& - \sum_{\ell=1}^{r} \int_J V_{t+\cdot}(\vartheta) \frac{\partial g_{t+\cdot}^\ell}{\partial \xi}(\vartheta) d\vartheta dW_t^\ell - V_t \int_J \frac{\partial f_{t+\cdot}}{\partial \xi}(\vartheta) U_{t+\cdot}(\vartheta) d\vartheta V_t dt \\
& + V_t \sum_{\ell=1}^{r} \left(\frac{\partial g_t^\ell}{\partial x} + \int_J \frac{\partial g_{t+\cdot}^\ell}{\partial \xi}(\vartheta) U_{t+\cdot}(\vartheta) d\vartheta V_t\right)^2 dt. \quad (2.30)
\end{aligned}
$$

Remark. (a)All the results which we have established in this work can be extended by replacing the Brownian motion W by another process $Z: [0, a] \times \Omega \to \mathbf{R}^{\mathbf{d}}$, $(d \in \mathbf{N})$ which is a continuous martingale adapted to $\{\mathcal{F}_t\}_{t\in[0,a]}$ and has independent increments and satisfies with some constant K the inequalities

$$\mathbf{E}[Z(t) - Z(s)]|\mathcal{F}_s| \leq K(t - s)\text{and}$$

$$\mathbf{E}(|Z(t) - Z(s)|^2|\mathcal{F}_s|) \leq K(t - s)\text{for}0 \leq s \leq t \leq T$$

Observe that the above properties of Z which we have just mentioned are the only properties of Wwhich we have used (in case of Brownian motion) to prove the results which we have obtained in this work.See (1) and (15).

(b) All the lemmas and theorems in this work hold for any delay interval $J' = [-r, 0)$ $(r \geq 0)$in placeof $J = [-1,0)$.See (1) and(15) .

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