On the Analysis of Matrix Geometric and Analytical Block NumericalIterative Methods for StationaryDistribution in the Structured Markov Chains

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Abstract
The modelled system is believed to have only one state at any given time, and its evolution is represented by transitions from one state to the next. This system's physical or mathematical behavior can also be depicted by defining all of the numerous states it can be in and demonstrating how it moves between them. In this study, the Markov chains whose transition matrices have a special block structure, a block structure that arises frequently when modeling queueing systems which leaves the transition matrices unchanged and saves time has been investigated, in order to provide some insight into the solutions of stationary distribution of Markov chain. Our quest is to compute the solutions using matrix geometric and analytical block numerical iterative methods on the Structured Markov Chains, by ensure that the matrix has the requisite block structure,the system of equations for π_0 and $\pi_1$have a solution and the normalization constant exist. Matrix operations are used with the help of some existing laws, theorems and formulas of Markov chain while the stationary distribution vector's $\pi_i,\ i = 1, 2, \ldots,$ are obtained for the illustrative examples.

Keywords: infinitesimal generator, logarithmic reduction algorithm, quasi-birth-death processes, random walk

1.0 Introduction

In the discipline of numerical analysis, there are two types of solution methods: iterative solution methods and direct solution methods. Iterative approaches start with an initial estimate of the solution vector and then alter it in such a way that it gets closer and closer to the genuine solution with each step or iteration. It eventually converges on the true solution.If there is no known initial approximation, a guess is performed or an arbitrary initial vector is used instead. The solution must be computed when a specified number of well-defined stages have been completed. The most widely utilized methods for deriving the stationary probability vector from either the stochastic transition probability matrix or the infinitesimal generator are iterative methods of one form or another.

This decision was made for a variety of reasons. First, a look at the conventional iterative approaches reveals that the matrices are only involved in one operation: multiplication with one or more vectors, which leaves the transition matrices unchanged. When the transition matrix is large and not banded, direct techniques are generally not preferred due to the volume of fill-in that can quickly overwhelm available storage capacity. Romanovsky (1970) established the application and simulation of discrete Markov Chains, which was followed by Stewart (1994, 2009) with the development of Numerical Solutions of Markov Chains, while Pesch*et al.* demonstrated the appropriateness of the Markov chain technique in the wind feed in Germany (2015).

Uzun and Kiral (2017) used the Markov chain model of fuzzy state to anticipate the direction of gold price movement and to estimate the probabilistic transition matrix of gold price closing returns, whereas Aziza *et al.* (2019) used the Markov chain model of fuzzy state to predict monthly rainfall data (2019). Clement (2019) demonstrated the application of Markov chain to the spread of disease infection, demonstrating that Hepatitis B became more infectious over time than tuberculosis and HIV, while Vermeer and Trilling (2020) demonstrated the application of Markov chain to journalism. However, in this study,the Analysis of Matrix Geometric and Analytical Block Numerical Iterative Methods for Stationary Distribution in the Structured Markov Chains is considered.

Notation

α arrival rate

μ departure rate

Q the infinitesimal generator

R rate matrix

θ normalization constant

2.0 Materials and Methods

An technique pioneered by Neuts examines the numerical solution of Markov chains whose transition matrices have a specific block structure, a block structure that emerges frequently when studying queueing systems (1981, 1989). These matrices have infinite block tridiagonal matrices in the simplest instance, with the three diagonal blocks repeating after some starting period. We create a matrix like this to represent this nonzero structure.

$$\begin{pmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0 & \cdots \\ B_{10} & A_1 & A_2 & 0 & 0 & 0 & \cdots \\ 0 & A_0 & A_1 & A_2 & 0 & 0 & \cdots \\ 0 & 0 & A_0 & A_1 & A_2 & 0 & \cdots \\ & & & \ddots & \ddots & \ddots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \qquad (1)$$

The submatrices A_0, A_1, and A_2 are all square and have the same dimension; the matrix B_{00} is also square but does not have to be the same size as A_1, and the dimensions of B_{01} and $B_{10}$are defined to be the same as B_{00} and A_1. When each state of the Markov chain can be represented as a pair $\{(\eta, k), \eta \geq 0, 1 \leq k \leq K\}$ and the states are ordered, first by increasing value of the parameter η, and then by increasing value of k for states with the same value, a transition matrix with this structure emerges.

This has the effect of categorizing the states into "levels" based on their relative importance.When transitions are permitted only between states of the same level (diagonal blocks), states in the next highest level (super-diagonal blocks), and states in the adjacent lower level, the block tridiagonal effect is achieved (sub-diagonal blocks). The blocks' repetitive nature arises when the transition rates probabilities are identical from level to level after boundary conditions are taken into account (which gives the initial blocks B_{00}, B_{01}, and B_{10}). A quasi-birth-death (QBD) process is defined as a Markov chain whose transition matrix has this block tridiagonal structure.

3.0 Results and Discussion

This section discusses the numerical examples and solutions on the analysis of matrix geometric and analytical block numerical iterative methods for stationary distribution in the structured Markov chains

3.1Illustrative example 1: Consider the Markov chain whose state transition diagram is shown below

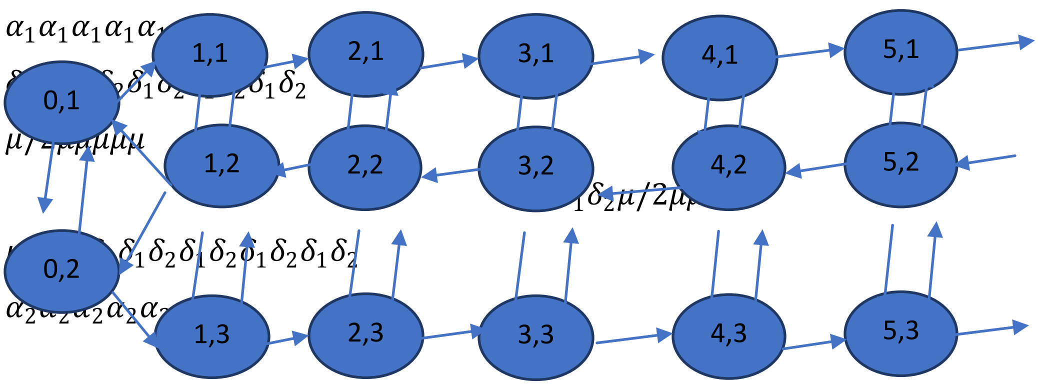

Figure 1. State transition diagram for an $M/M/1$-type process.

The transition matrix is

$$Q=\left(\begin{array}{cc|ccc|ccc|ccc|ccc|c}
* & \delta_1 & \alpha_1 & & & & & & & & & & & & \\
\delta_2 & * & & & \alpha_2 & & & & & & & & & & \\
\hline
 & & * & \delta_1 & & \alpha_1 & & & & & & & & & \\
\mu/2 & \mu/2 & \delta_2 & * & \delta_1 & & & & & & & & & & \\
 & & & \delta_2 & * & & & \alpha_2 & & & & & & & \\
\hline
 & & & & & * & \delta_1 & & \alpha_1 & & & & & & \\
 & & & \mu & & \delta_2 & * & \delta_1 & & & & & & & \\
 & & & & & & \delta_2 & * & & & \alpha_2 & & & & \\
\hline
 & & & & & & & & * & \delta_1 & & \alpha_1 & & & \\
 & & & & & & \mu & & \delta_2 & * & \delta_1 & & & & \\
 & & & & & & & & & \delta_2 & * & & & & \\
\hline
 & & & & & & & & & & & * & \delta_1 & & \\
 & & & & & & & & & \mu & & \delta_2 & * & \delta_1 & \\
 & & & & & & & & & & & & \delta_2 & * & \\
\hline
 & & & & & & & & & & & & & \ddots & \ddots
\end{array}\right)$$

It has the characteristic block tridiagonal structure, making it an excellent candidate for numerical solution using the approaches proposed in this paper. Its diagonal elements, denoted by asterisks (*), have a sum of zero across each row. The following block matrices are obtainable:

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 0 \end{pmatrix},\ A_1 = \begin{pmatrix} -(\delta_1+\alpha_1) & \delta_1 & 0 \\ \delta_2 & -(\mu+\delta_1+\delta_2) & \delta_1 \\ 0 & \delta_2 & -(\delta_2+\alpha_2) \end{pmatrix},\ A_2 = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha_2 \end{pmatrix},$$

(2)

and

$$B_{00} = \begin{pmatrix} -(\delta_1+\alpha_1) & \delta_1 \\ \delta_2 & -(\delta_2+\alpha_2) \end{pmatrix},\quad B_{01} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_2 \end{pmatrix},\quad B_{10} = \begin{pmatrix} 0 & 0 \\ \mu/2 & \mu/2 \\ 0 & 0 \end{pmatrix}$$

(3)

3.2 The Quasi-Birth-Death(QBD) Case

The matrix geometric method can be used to solve quasi-birth-death processes efficiently and simply. We start by looking at the case where a QBD process' blocks are reduced to single elements. Consider the infinite infinitesimal generator described below:

$$Q = \begin{pmatrix} -\alpha & \alpha & & & & \\ \mu & -(\alpha+\mu) & \alpha & & & \\ & \mu & -(\alpha+\mu) & \alpha & & \\ & & \mu & -(\alpha+\mu) & \alpha & \\ & & & \ddots & \ddots & \ddots \end{pmatrix} \qquad (4)$$

This could be related to the random walk issue, where the probability of moving from any state k to state $(k + 1)$ is $\frac{\alpha}{(\alpha+\mu)}$ and the likelihood of going from state k to state $(k - 1)$ is $\frac{\mu}{(\alpha+\mu)}$.. We can write $-\alpha\pi_0 + \mu\pi_1 = 0$,from $\pi Q = 0$, which yields

$$\pi_1 = \frac{\alpha}{\mu}\pi_0,$$

and in general

$$\alpha\pi_{i-1} - (\alpha+\mu)\pi_i + \mu\pi_{i+1} = 0.$$

To demonstrate that, we use induction.

$$\pi_{i+1} = \frac{\alpha}{\mu}\pi_i\ ,\ \ for\ i = 1, 2, \ldots$$

Since

$$\pi_1 = \frac{\alpha}{\mu}\pi_0,$$

We can deduce the following from the inductive hypothesis:

$$\pi_i = \frac{\alpha}{\mu}\pi_{i-1}\,,\ \ for\ i = 1, 2, \ldots$$

And hence

$$\pi_{i+1} = \frac{(\alpha+\mu)}{\mu}\pi_i - \left(\frac{\alpha}{\mu}\right)\pi_{i-1} = \left(\frac{\alpha}{\mu}\right)\pi_{i,}$$

This is the desired outcome. It is clearly clear that

$$\pi_i = \left(\frac{\alpha}{\mu}\right)^i \pi_0 = \rho^i \pi_0, \qquad (5)$$

where $\rho = \left(\frac{\alpha}{\mu}\right)$, Once π_0 has been determined, the remaining values, π_i, $i = 1, 2,$ ..., can be determined recursively. When Q is a QBD process, the result is similar: In this context, the parameter is a square matrix R of order K and the stationary distribution's components π_i are sub-vectors of length K. In a QBD process, let Q be the infinitesimal generator. Then

$$Q = \begin{pmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0 & \cdots \\ B_{10} & A_1 & A_2 & 0 & 0 & 0 & \cdots \\ 0 & A_0 & A_1 & A_2 & 0 & 0 & \cdots \\ 0 & 0 & A_0 & A_1 & A_2 & 0 & \cdots \\ & & & \ddots & \ddots & \ddots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

and the stationary distribution is obtained from $\pi Q = 0$. Let πbe partitioned conformally with Q, i.e.,

$$\pi = (\pi_0,\ \pi_1,\ \pi_2,\ \ldots,)$$

Where

$$\pi_i = \big(\pi(i,1),\ \pi(i,2), \ldots,\ \pi(i,K)\big)$$

$\pi(i,k)$ is the probability of finding the system in state (i,k) in steady state for $i = 0, 1, \ldots$,

As a result, the following equations emerge:

$$\pi_0 B_{00} + \pi_1 B_{10} = 0,$$

$$\pi_0 B_{01} + \pi_1 A_1 + \pi_2 A_0 = 0,$$

$$\pi_1 A_2 + \pi_2 A_1 + \pi_3 A_0 = 0,$$

$$\vdots$$

$$\pi_{i-1}A_2 + \pi_i A_1 + \pi_{i+1}A_0 = 0, \;\; i = 1, 2, ..., \qquad (6)$$

It may be proven, similarly to the point situation, that there exists a constant matrix R such that

$$\pi_i = \pi_{i-1}R, \quad for \;\; i = 2, 3, ..., \qquad (7)$$

Since the sub-vectors π_i are geometrically connected to each other, they are said to be geometrically related.

$$\pi_i = \pi_1 R^{i-1}, \;\; i = 2, 3, ..., \qquad (8)$$

Equation may be used to create the remaining sub-vectors of the stationary distribution if the sub-vectors π_0 and π_1 and the rate matrix R can be found using Equation (7). Returning to

$$\pi_{i-1}A_2 + \pi_i A_1 + \pi_{i+1}A_0 = 0, \;\; i = 1, 2, ...,$$

We get for $i = 2, 3, ...$, by inserting from Equation (8).

$$\pi_1 R^{i-2}A_2 + \pi_1 R^{i-1}A_1 + \pi_1 R^i A_0 = 0,$$

i.e.,

$$\pi_1 R^{i-2}(A_2 + RA_1 + R^2 A_0) = 0.$$

R can now be calculated from

$$(A_2 + RA_1 + R^2 A_0) = 0. \qquad (9)$$

successive substitution is the simplest way to accomplish this. We can deduce Equation (9) by applying the assumption that A_1 must be nonsingular.

$$A_2 A_1^{-1} + R + R^2 A_0 A_1^{-1} = 0,$$

i.e.,

$$R = -A_2 A_1^{-1} - R^2 A_0 A_1^{-1} = -V - R^2 W,$$

Where $V = -A_2 A_1^{-1}$ and $W = A_0 A_1^{-1}$.As a result, Neuts(1981, 1989), proposes the successive substitution approach namely

$$R_0 = 0 \quad \text{and} \quad R_{(k+1)} = -V - R_k^2 W, \;\; k = 0, \; 1, \; \dots, \quad (10)$$

The series of matrices $R_{(k)}, k = 0, 1, 2, ...$ as proven by Neuts, is nondecreasing and converges to the rate matrix R. Once subsequent differences are smaller than a defined tolerance criterion, the process is stopped. Unfortunately, this straightforward strategy has the drawback of necessitating a large number of repetitions before obtaining a sufficiently accurate matrix R. Latouche and Ramaswami (1994) created a logarithmic reduction technique that achieves extraordinarily fast quadratic convergence (the number of decimal places doubles at each iteration).There are several cases when the rate matrix R can be generated explicitly without the need for any iterations at all. The derivation of π_0 and π_1 is the only obvious problem. The first two $\pi Q = 0$equations are

$$\pi_0 B_{00} + \pi_1 B_{10} = 0,$$

$$\pi_0 B_{01} + \pi_1 A_1 + \pi_2 A_0 = 0,$$

We obtain by replacing$\pi_2$with $\pi_1 R$ and expressing these equations in matrix form.

$$(\pi_0 \quad \pi_1)\begin{pmatrix} B_{00} & B_{01} \\ B_{10} & A_1 + RA_0 \end{pmatrix} = (0 \quad 0) \qquad (11)$$

This system can be solved to produce π_0 and π_1 using the rate matrix R and the blocks B_{00}, B_{01}, B_{10}, A_1, and A_0,. Due to the fact that this is a homogeneous system of equations, the computed solution must be normalized so that the components of π sum to one. To put it another way, we insist on $\pi e = 1$. Thus

$$1 = \pi e = \pi_0 e + \pi_1 e + \sum_{i=2}^{\infty} \pi_i e$$

$$= \pi_0 e + \pi_1 e + \sum_{i=2}^{\infty} \pi_1 R^{i-1} e$$

$$= \pi_0 e + \sum_{i=1}^{\infty} \pi_1 R^{i-1} e = \pi_0 e + \sum_{i=0}^{\infty} \pi_1 R^i e.$$

This implies the condition

$$\pi_0 e + \pi_1 \left(\sum_{i=0}^{\infty} R^i \right) e = 1.$$

R's eigenvalues are all within the unit circle, implying that (I - R) is nonsingular.

$$\left(\sum_{i=0}^{\infty} R^i\right) = (1 - R)^{-1}. \qquad (12)$$

By computing the normalization of the vectors π_0 and π_1, we would complete the normalization.

$$\theta = \pi_0 e + \pi_1 (1 - R)^{-1} e$$

and dividing the computed sub-vectors π_0 and π_1 by θ.

When B_{00} is the same size as the A blocks and $B_{01} = A_2, \pi_i = \pi_0 R^i, for\ i = 1, 2, ...,$. is the simplest example. In this situation, the system of equations (11) can be substituted with the simpler system,$\pi_0(B_{00} + RB_{10}) = 0$, from which $\pi_0(1 - R)^{-1} e = 1$ can be derived and then normalized to obtained π. The Markov chain is positive recurrent only if the probability of advancing to higher-numbered states is strictly less than the chance of moving to lower-numbered states in a random walk issue.For a QBD process to be ergodic, the drift to higher-numbered levels must be strictly less than the drift to lower-numbered levels. Let πA be the stationary distribution of the infinitesimal generator $A = A_0 + A_1 + A_2$. The following conditions must be met for a QBD process to be ergodic:

$$\pi_A A_2 e < \pi_A A_0 e \qquad (13)$$

Note that the elements of A_2 move the process up a level, while the elements of A_0 move it down. Indeed, Neuts proves that the spectral radius of R is strictly less than 1, and hence that the matrix $1 - R$ is nonsingular, based on this requirement. It suffices to substitute $-A_1^{-1}$ with$(1 - A1-1$ when the Markov chain under examination is a discrete-time Markov chain with a stochastic transition probability matrix. However, it's worth noting that A_1 is the repeating diagonal block in a transition rate matrix in the first case, and the repeating diagonal block in a stochastic matrix in the second.The equations are similar; the values in the blocks differ depending on whether the global matrix is an infinitesimal generator or a stochastic matrix. The following are the steps that must be followed when using the matrix geometric approach to solve a QBD process:

i. Ensure that the matrix has the requisite block structure.

ii. Use Equation (13) to ensure that the Markov chain is ergodic.

iii. Use Equation (10) to compute the matrix *R*.

iv. Solve the system of equations (11) for π_0 and π_1.

v. Compute the normalizing constant $\theta$and normalize π_0 and π_1.

vi. Use Equation (7) to compute the remaining components of the stationary distributionvector.

3.3 Illustrative example 2: We'll use the matrix geometric approach to solve the Markov chain in Example 1 with the following parameter values:

$$\alpha_1 = 1, \alpha_2 = 0.5, \mu = 4, \delta_1 = 5, \delta_2 = 3,$$

The infinitesimal generator is then given by

$$Q=\begin{pmatrix}
-6 & 5 & & 1 & & & & & & & \\
3 & -3.5 & & & 0.5 & & & & & & \\
& & -6 & 5 & & 1 & & & & & \\
2 & 2 & 3 & -12 & 5 & & & & & & \\
& & & 3 & -3.5 & & & 0.5 & & & \\
& & & & & -6 & 5 & & 1 & & \\
& & & & & 3 & -12 & 5 & & & \\
& & & & & & 3 & -3.5 & & & 0.5 \\
& & & & & \ddots & & & \ddots & & \ddots
\end{pmatrix}$$

$$A_0=\begin{pmatrix}0 & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & 0\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$A_1=\begin{pmatrix}-(\delta_1+\alpha_1) & \delta_1 & 0\\ \delta_2 & -(\mu+\delta_1+\delta_2) & \delta_1\\ 0 & \delta_2 & -(\delta_2+\alpha_2)\end{pmatrix}=\begin{pmatrix}-6 & 5 & 0\\ 3 & -12 & 5\\ 0 & 3 & -3.5\end{pmatrix},$$

$$A_2=\begin{pmatrix}\alpha_1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \alpha_2\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0.5\end{pmatrix},$$

$$B_{00}=\begin{pmatrix}-(\delta_1+\alpha_1) & \delta_1\\ \delta_2 & -(\delta_2+\alpha_2)\end{pmatrix}=\begin{pmatrix}-6 & 5\\ 3 & -3.5\end{pmatrix},$$

$$B_{01}=\begin{pmatrix}\alpha_1 & 0 & 0\\ 0 & 0 & \alpha_2\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0.5\end{pmatrix},$$

$$B_{10}=\begin{pmatrix}0 & 0\\ \mu/2 & \mu/2\\ 0 & 0\end{pmatrix}=\begin{pmatrix}0 & 0\\ 2 & 2\\ 0 & 0\end{pmatrix}.$$

i.The matrix obviously has the correct QBD structure.

ii. We check that the system is stable by verifying Equation (13). The infinitesimal generator matrix

$$A = A_0 + A_1 + A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -6 & 5 & 0 \\ 3 & -12 & 5 \\ 0 & 3 & -3.5 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.5 \end{pmatrix}$$

$$A = \begin{pmatrix} -5 & 5 & 0 \\ 3 & -8 & 5 \\ 0 & 3 & -3 \end{pmatrix}$$

∴ A has stationary probability vector

$$\pi_A = (0.1837 \quad 0.3061 \quad 0.5102)$$

and

$$\pi_A A_2 e = (0.1837 \quad 0.3061 \quad 0.5102) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.5 \end{pmatrix} e = 0.4388e$$

$$\pi_A A_0 e = (0.1837 \quad 0.3061 \quad 0.5102) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} e = 1.2244e$$

$$0.4388e = \pi_A A_2 e < \pi_A A_0 e = 1.2244e$$

iii. We now initiate the iterative procedure to compute the rate matrix R. The inverse of A_1 is

$$A_1^{-1} = \begin{pmatrix} -0.2466 & -0.1598 & -0.2283 \\ -0.0959 & -0.1918 & -0.2740 \\ -0.0822 & -0.1644 & -0.5205 \end{pmatrix}$$

which allows us to compute

$$V = A_2 A_1^{-1} = \begin{pmatrix} -0.2466 & -0.1598 & -0.2283 \\ 0 & 0 & 0 \\ -0.0411 & -0.0822 & -0.2603 \end{pmatrix}$$

and

$$W = A_0 A_1^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ -0.3836 & -0.7671 & 1.0959 \\ 0 & 0 & 0 \end{pmatrix}$$

Equation (10) becomes

$$R_{(k+1)} = \begin{pmatrix} 0.2466 & 0.1598 & 0.2283 \\ 0 & 0 & 0 \\ 0.0411 & 0.0822 & 0.2603 \end{pmatrix} + R_k^2 \begin{pmatrix} 0 & 0 & 0 \\ 0.3836 & 0.7671 & 1.0959 \\ 0 & 0 & 0 \end{pmatrix}$$

and iterating successively, beginning with $R_{(0)} = 0$, we find

$$R_{(1)} = \begin{pmatrix} 0.2466 & 0.1598 & 0.2283 \\ 0 & 0 & 0 \\ 0.0411 & 0.0822 & 0.2603 \end{pmatrix},$$

$$R_{(2)} = \begin{pmatrix} 0.2689 & 0.2044 & 0.2921 \\ 0 & 0 & 0 \\ 0.0518 & 0.1036 & 0.2909 \end{pmatrix}$$

$$R_{(3)} = \begin{pmatrix} 0.2793 & 0.2252 & 0.3217 \\ 0 & 0 & 0 \\ 0.0567 & 0.1134 & 0.3049 \end{pmatrix}$$

$$\vdots$$

As predicted by Neuts(1981), the elements do not decrease in number. After 48 iterations, the differences between subsequent iterations are less than 10^{-12}, at which point

$$R_{(48)} = \begin{pmatrix} 0.2917 & 0.2500 & 0.3571 \\ 0 & 0 & 0 \\ 0.0625 & 0.1250 & 0.3214 \end{pmatrix}$$

iv. By proceeding to the boundary condition

$$(\pi_0 \quad \pi_1)\begin{pmatrix} B_{00} & B_{01} \\ B_{10} & A_1 + RA_0 \end{pmatrix} = (\pi_0 \quad \pi_1)\begin{pmatrix} -6 & 5 & 1 & 0 & 0 \\ 3 & -3.5 & 0 & 0 & 0.5 \\ 0 & 0 & -6 & 6 & 0 \\ 2 & 2 & 3 & -12 & 5 \\ 0 & 0 & 0 & 3.5 & -3.5 \end{pmatrix} = (0 \quad 0)$$

We can solve this by substituting $\pi_{0_1} = 1$ for the last equation, i.e. setting the first component of the sub-vector π_0 to 1. The equations become a system of equations.

$$(\pi_0 \quad \pi_1)\begin{pmatrix} -6 & 5 & 1 & 0 & 0 \\ 3 & -3.5 & 0 & 0 & 0.5 \\ 0 & 0 & -6 & 6 & 0 \\ 2 & 2 & 3 & -12 & 5 \\ 0 & 0 & 0 & 3.5 & -3.5 \end{pmatrix} = (0 \quad 0 \mid \quad 0 \quad 0 \quad 1)$$

with solution

$$(\pi_0 \quad \pi_1) = (1 \quad 1.6923 \mid \quad 0.3974 \quad 0.4615 \quad 0.9011)$$

v. The normalization constant is

$$\theta = \pi_0 e + \pi_1 (1 - R)^{-1} e$$

$$\theta = (1 \quad 1.6923)e + (0.3974 \quad 0.4615 \quad 0.9011)\begin{pmatrix} 1.4805 & 0.4675 & 0.7792 \\ 0 & 1 & 0 \\ 0.1364 & 0.2273 & 0.1546 \end{pmatrix} e$$

$$= 2.6923 + 3.2657 = 5.9580$$

which allows us to compute

$$\frac{\pi_0}{\theta} = (0.1678 \quad 0.2840) \quad \text{and} \quad \frac{\pi_1}{\theta} = (0.0667 \quad 0.0775 \quad 0.1512).$$

vi. Successive subcomponents of the stationary distribution are now computed from

$$\pi_i = \pi_{i-1}R, \quad for \quad i = 2, 3, \ldots,.$$

For example,

$$\pi_2 = \pi_1 R = (0.0667 \quad 0.0775 \quad 0.1512)\begin{pmatrix} 0.2917 & 0.2500 & 0.3571 \\ 0 & 0 & 0 \\ 0.0625 & 0.1250 & 0.3214 \end{pmatrix}$$

$$= (0.0289 \quad 0.0356 \quad 0.0724)$$

$$\pi_3 = \pi_2 R = (0.0289 \quad 0.0356 \quad 0.0724)\begin{pmatrix} 0.2917 & 0.2500 & 0.3571 \\ 0 & 0 & 0 \\ 0.0625 & 0.1250 & 0.3214 \end{pmatrix}$$

$$= (0.0130 \quad 0.0356 \quad 0.0336),$$

4.0 Conclusion

the iterative solution methods for the stationary distribution of Markov chains which start with an initial estimate of the solution vector and then alter it in such a way that it gets closer and closer to the genuine solution with each step or iterationis examined on the Markov chains whose transition matrices have a special block structure, a block structure that arises frequently when modeling queueing systems which leaves the transition matrices unchanged and saves time has been investigated, in order to provide some insight into the solutions of stationary distribution of Markov chain. Our quest is to compute the solutions using matrix geometric and analytical block numerical iterative methods on the Structured Markov Chains by considered the following steps: Ensure that the matrix has the requisite block structure,Solving the system of equations for π_0 and π_1 and the normalization constant. Matrix operations are used with the help of some existing laws, theorems and formulas of Markov chain while the stationary distribution vector's $\pi_i, \; i = 1, 2, \ldots,$ are obtained for the illustrative examples.

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